Spinor representation of surfaces and complex stresses on membranes and interfaces

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Abstract

Variational principles are developed within the framework of a spinor representation of the surface geometry to examine the equilibrium properties of a membrane or interface. This is a far-reaching generalization of the Weierstrass-Enneper representation for minimal surfaces, introduced by mathematicians in the nineties, permitting the relaxation of the vanishing mean curvature constraint. In this representation the surface geometry is described by a spinor field, satisfying a two-dimensional Dirac equation, coupled through a potential associated with the mean curvature. As an application, the mesoscopic model for a fluid membrane as a surface described by the Canham-Helfrich energy quadratic in the mean curvature is examined. An explicit construction is provided of the conserved complex-valued stress tensor characterizing this surface.

1 Introduction

The coarse-grained description of a physical surface is often entirely insensitive to microscopic details, involving an energy function that depends only on the surface geometry. A familiar example is provided by a soap film or an interface which will reach equilibrium when its surface area is minimized. In the absence of a pressure difference across the surface, this state is described by a minimal surface characterized by a vanishing mean curvature; it will thus assume a saddle shape almost everywhere.

From a mathematical point of view minimal surfaces are very special: the functions embedding the surface in space are harmonic so that, in an appropriate parametrization, the surfaces will be described by analytic functions. This is the essential element of the classical Weierstrass-Enneper (WE) representation of minimal surfaces, introduced 150 years ago, which has played a central role in the development of the subject ever since [1, 2]. While physically it is often adequate to settle for a small gradient approximation in terms of a height function, where one can get by with a more modest toolkit, the WE representation of the surface geometry provides an indispensable handle on the global features of minimal surfaces; the Schwartz P Surface or plumber's nightmare [3], describing liquid crystalline phases of fluid membranes, provides a good case in point [4].

Even if one is interested in interfaces, however, one almost always needs to accommodate a pressure difference across this surface; the minimal surface is replaced by one of constant mean curvature. Never mind addressing questions of stability or examining fluctuations, simple as this modification may be, its description lies beyond the scope of the WE representation.

There is, however, a generalization of the WE framework that applies to any surface. Back in 1979 Kenmotsu showed how the classical representation could be tweaked to describe surfaces with any prescribed mean curvature [5]; a decade later, the WE representation was reformulated in terms of a two-component spinor field, which also turned out to be its natural setting [6, 7]. By the mid-nineties, this framework had been extended to accommodate a non-vanishing mean curvature [8, 9]. As emphasized by Konopelchenko and Taimanov in [10], it was no longer even appropriate to think of the mean curvature as prescribed. For reviews see references [11] and [12].

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The spinor appearing in this framework satisfies a linear Dirac-type equation on the complex plane; its components couple through a real-valued external potential. As in the classical WE representation, the functions which describe the embedding of the surface in space occur as integrals of closed differential one-forms quadratic in the spinor field. The mean curvature of the surface is proportional to the potential. When it vanishes, the components uncouple and the original representation for minimal surfaces is recovered.

Our focus will be on the mathematical description of fluid membranes on mesoscopic scales [13]. On these scales the membrane is described with unusual accuracy as a two dimensional surface; the energy associated with a given configuration is the Canham-Helfrich bending energy quadratic in the mean curvature [14]. Unlike any familiar elastic material, in-plane shear goes unpenalized; thus the surface is characterized completely by its geometrical degrees of freedom and membrane elasticity can be addressed in terms of geometry. A striking feature of this energy in the spinor representation is that the integrated mean curvature squared depends only of the potential. Modulo a topological contribution to the energy, this is also the most general geometrical energy associated with bending. It is thus curious that there has been no systematic attempt made to look more closely at this functional in terms of these variables. In this paper we will examine the behavior of the Canham-Helfrich energy under variations of the potential; in particular we will show how the equation describing equilibrium shapes–non-linear both in the spinor and in the potential–is obtained.

What would appear to be the obvious way to do this also turns out to be riven with technical difficulties: if the spinors and the potential are treated as the fundamental variables, the relationship connecting their variations to that of the embedding functions defining the surface is non-local. One way to sidestep this difficulty is to introduce an appropriate set of Lagrange multipliers, a spinor analogue of a framework introduced a few years ago to reformulate the variational principles for parametrized surfaces in terms of the induced metric and the extrinsic curvature [15]. Just as the metric and the extrinsic curvature are not independent variables, the spinor and the potential are also constrained (by the Dirac equation). This is just as well; for a naive variation of the potential appears to imply that the only critical points of bending energy are minimal surfaces, a manifestly incorrect conclusion. In this framework, the relationship connecting the spinors to a surface, characterized by its three embedding functions, will also be input as a constraint. The multiplier enforcing this constraint quantifies the local change in energy as the surface is deformed; it is thus identified as a "stress tensor". Unlike the stress in a typical elastic medium, however, it will be determined completely by the geometry. A surface in mechanical equilibrium is described by a conserved stress tensor.

The description of equilibrium in terms of a geometrical stress tensor has been discussed previously in the context of a parametrized description of the surface; in reference [16] it was identified as the Noether current associated with the translational invariance of the energy, an approach refined in reference [15] using the framework of auxiliary variables. More physical continuum mechanical/thermodynamical treatments have been presented in references [17] and [18]). Work in the latter direction dates back to the work of Evans [19]. The concept of geometrical surface stress is also familiar, if only implicitly, in the context of minimal surfaces [20, 21] where they appear in the identification of the weights–external forces–that provide the tension preventing the collapse of the surface.

A problem of significant current biophysical interest is the study of interactions between surface-bound particles mediated by the surface geometry, with the particles in question identified as proteins (see reference [22], for example, and more recently [23]). In [24] this problem was approached in the non-linear regime by examining the geometrical stresses associated with the deformed geometry. By casting the Euler-Lagrange equations as a conservation law it became possible to gain access to global information which is not manifest when the divergence is dismantled into tangential and normal parts. Just as the techniques of complex variables can be exploited very effectively in the linear regime (as demonstrated spectacularly in reference [22]), the spinor framework renders it possible to apply these same tools to examine physical processes which lie beyond the scope of a linear description. Recent reviews, which summarize nicely the role of geometrical stresses in a soft matter and biophysical context, have been provided by Deserno [25] and Powers [26].

The simplification of the surface geometry in this representation involves giving up, to a large degree, the reparametrization invariance inherent in the geometrical nature of the problem. An arbitrary deformation of the surface will not be compatible with the manifestly conformally flat form of the metric. For example,

any purely normal deformation of a non-planar surface will generally be inconsistent with the representation; consistency will require a compensating tangential deformation to tag along. Fortunately, this awkward technical point never needs to be addressed explicitly in the variational framework we introduce. It does, however, manifest itself in the conservation law for the stress tensor. Whereas the tangential conservation laws would be expected to be satisfied identically in any completely reparametrization invariant framework, where small tangential deformations of the geometrical degrees of freedom are identified with reparametrizations (see [27]), here they are not. This is also, of course, just as well: the Euler-Lagrange equations for the spinor and the potential leave undetermined the off-diagonal component of the tangential stress; the tangential components of the conservation law turn out to provide the differential equations determining this missing component.

It is evident that the description of equilibrium in terms of a conserved stress tensor possesses a number of interesting model-independent properties that are best appreciated by extending the discussion to accommodate "energy" functionals of a more general kind. We will thus examine geometrical energies involving powers of the mean curvature. The Canham-Helfrich model occurs as a special case.

We begin in section 2 with a brief review of relevant features of the spinor representation of surfaces. Our main results are contained in section 3 where variational principles are developed within the spinor framework. In particular, the energy most relevant in condensed matter applications, consisting of a linear combination of area, the integrated mean curvature and bending energy is considered. We conclude with a brief discussion as well as a few suggestions for future work. Various useful identities are collected in a set of appendices.

2 Dirac equation for surface spinors

In the generalized WE representation of the surface Σ the three functions describing its embedding in three-dimensional Euclidean space $\mathbb{E}^3$ are expressed in terms of a 2-component spinor field $\psi(z,\bar z) = (\psi_1(z,\bar z), \psi_2(z,\bar z))^T$ defined on a simply connected domain D of the complex plane $\mathbb{C}$ (the overbar denotes the complex conjugate). This spinor is a solution of the Dirac equation

$$\mathcal{D}\psi = 0\,, \tag{1}$$

where $\mathcal{D}$ is the first order differential operator,

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{pmatrix}, \tag{2}$$

involving a real valued potential $\mathcal{V}$. This potential couples the spinor components; Eq.(1) reads

$$\partial_z \psi_2 = -\mathcal{V}\psi_1, \quad \partial_{\bar z}\psi_1 = \mathcal{V}\psi_2\,. \tag{3}$$

The spinor ξ defined by $\xi = (-\bar\psi_2, \bar\psi_1)^T$ also satisfies the Dirac equation:

$$\mathcal{D}\xi = 0\,. \tag{4}$$

We will also write $\xi = \zeta\bar\psi$, where

$$\zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Note also that ξ unlike $\bar\psi$ transforms like ψ under the action of $SU(2)$. A linear combination of ψ and ζ corresponds to a rotation of the surface, as described in Appendix A.

The immersion of D into $\mathbb{E}^3$ defined by $\psi(z,\bar z)$ is given by

$$\mathbf{X}(z,\bar z) = \int_\gamma \boldsymbol{\phi}(w,\bar w)\,, \tag{6}$$

where $\boldsymbol{\phi}(z,\bar z)$ is the vector-valued 1-form with the following components[1]

$$\begin{aligned} \phi^1(z,\bar z) &= Re\left[(\bar\psi_2{}^2 - \psi_1{}^2)dz\right], & (7a)\\ \phi^2(z,\bar z) &= Re\left[i(\bar\psi_2{}^2 + \psi_1{}^2)dz\right], & (7b)\\ \phi^3(z,\bar z) &= 2Re\left[\psi_1\bar\psi_2 dz\right]\;; & (7c)\end{aligned}$$

[1] Here we interchange ϕ^1 and ϕ^2 with respect to the more usual definitions in the literature, for example [12], as this facilitates the connection with the classical WE representation. The number of minus signs appearing in the calculations is also reduced.

γ is a path on the complex plane terminating at the point z. It is evident from their definition that the embedding functions are invariant under complex conjugation, $\psi \to \bar{\psi}$.

It is simple to show that the 1-forms (7) are closed in the complex plane, i.e. $d\phi^i = 0$: one uses the fact that the spinor ψ satisfies the Dirac equation, (1) and that $\mathcal{V}$ is real. Note that the derivatives $\partial_z \psi_1$ and $\partial_{\bar{z}} \psi_2$ never need to be evaluated in this argument; in addition, it should be remarked that the corresponding 1-forms constructed with the imaginary part in Eq.(7) are not closed.

Closure and Stokes theorem together imply that the definition of the embedding functions (6) is independent of the choice of the path γ and thus well defined. It is worth pointing out that the Dirac equation (1) appears to be the most general linear equation for the spinor consistent with closure.

2.1 Intrinsic geometry of the surface

The tangent vectors to the surface adapted to this parametrization are $\mathbf{e}_z = \partial_z \mathbf{X}$ and $\mathbf{e}_{\bar{z}} = \bar{\mathbf{e}_z}$; in terms of the spinor components $\mathbf{e}_z$ is given by

$$\mathbf{e}_z = \frac{1}{2} \begin{pmatrix} \bar{\psi_2}^2 - {\psi_1}^2 \\ i\left(\bar{\psi_2}^2 + {\psi_1}^2\right) \\ 2\psi_1 \bar{\psi}_2 \end{pmatrix} . \tag{8}$$

The vector-valued 1-form $\boldsymbol{\phi}$ defined in Eq.(7) can thus be expressed in the alternative form $\boldsymbol{\phi} = 2Re\left(\mathbf{e}_z dz\right)$.

The two tangent vectors are null with respect to the scalar product in $\mathbb{E}^3$ (denoted by $\cdot$), in other words, their norms vanish: $\mathbf{e}_z \cdot \mathbf{e}_z = 0 = \mathbf{e}_{\bar{z}} \cdot \mathbf{e}_{\bar{z}}$. However they are not orthogonal: $\mathbf{e}_z \cdot \mathbf{e}_{\bar{z}} = 1/2\,|\psi|^4$, where $|\psi|^2 = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2$ and $|\psi_i|^2 = \bar{\psi}_i \psi_i$.

This parametrization is isothermal (or conformal): the line element is given by

$$ds^2 = |\psi|^4 |dz|^2 ; \tag{9}$$

the induced metric on the surface assumes the form

$$g_{ab} = \frac{|\psi|^4}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{10}$$

So the metric tensor is a Weyl rescaling of its Euclidean counterpart on the complex plane, i.e., it is manifestly conformally flat. The fourth power of the spinor norm provides the conformal factor. The intrinsic geometry of the surface is completely determined by $|\psi|$.

The residual reparametrization freedom consistent with the isothermal form of the metric is captured by analytic functions. Under the holomorphic transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$, one finds that

$$\psi_1(z,\bar{z}) \to w'(z)^{1/2} \psi_1(w,\bar{w}) , \quad \psi_2(z,\bar{z}) \to \bar{w}'(\bar{z})^{1/2} \psi_2(w,\bar{w}) , \quad \mathcal{V}(z,\bar{z}) \to |w'(z)|\, \mathcal{V}(w,\bar{w}) . \tag{11}$$

These variables are thus scalar densities.

The determinant of the metric is $g = -1/4|\psi|^8$, so that $\sqrt{g} = i/2|\psi|^4$ is imaginary. The area is given by

$$A = \frac{i}{2} \int dz \wedge d\bar{z}\, |\psi|^4 . \tag{12}$$

Despite appearances, it is a real form: $d\bar{A} = -i/2|\psi|^4\, d\bar{z} \wedge dz = dA$.

2.2 Extrinsic geometry of the surface

The normal vector to the surface Σ defined by $\mathbf{n} = \mathbf{e}_z \times \mathbf{e}_{\bar{z}} / \sqrt{g}$, in terms of the spinor components is given by

$$\mathbf{n} = \frac{1}{|\psi|^2} \begin{pmatrix} 2\,Re\,(\psi_1 \psi_2) \\ 2\,Im\,(\psi_1 \psi_2) \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix} . \tag{13}$$

The second fundamental form or extrinsic curvature tensor defined by $K_{ab} = -\mathbf{n}\cdot\partial_a\mathbf{e}_b$, is given by

$$K_{ab} = \begin{pmatrix} \mathcal{A} & \mathcal{V}|\psi|^2 \\ \mathcal{V}|\psi|^2 & \bar{\mathcal{A}} \end{pmatrix}, \tag{14}$$

where $\mathcal{A}$ denotes the Wronskian of ψ_1 and $\bar{\psi}_2$, $\mathcal{A} = \bar{\psi}_2\partial_z\psi_1 - \psi_1\partial_z\bar{\psi}_2$. $\mathcal{A}$ is invariant under rotation, as are the norm of ψ and $\mathcal{V}$. Under reparametrization, $z \to w(z)$, $\mathcal{A}$ transforms as a scalar density:

$$\mathcal{A}(z,\bar{z}) \to w'(z)^2\mathcal{A}\,. \tag{15}$$

Besides the potential and the spinor norm, the Wronskian is the other function of the spinor and its derivatives which occurs naturally in this framework.

The eigenvalues of the shape operator $K^a{}_b = g^{ac}K_{cb}$ (g^{ab} is the inverse metric) are the two principal curvatures, C_1 and C_2. The two symmetric curvature invariants, (twice) the mean curvature $K = C_1 + C_2$ and the Gaussian curvature $K_G = C_1C_2$ are given respectively by

$$K = \frac{4}{|\psi|^2}\,\mathcal{V}, \tag{16}$$

$$K_G = \frac{4}{|\psi|^8}\left(\mathcal{V}^2|\psi|^4 - |\mathcal{A}|^2\right). \tag{17}$$

If $\mathcal{V} = 0$ the surface is minimal ($K = 0$). By defining the components of the spinor ψ in terms of two analytic functions $f(z)$ and $g(z)$ as $\psi_1 = f^{1/2}g/\sqrt{2}$ and $\psi_2 = \bar{f}^{1/2}/\sqrt{2}$ we obtain the $\boldsymbol{\phi}$ corresponding to the original WE representation for minimal surfaces [2] given by

$$\boldsymbol{\phi} = Re\left[\frac{f}{2}\left(1-g^2, 1+g^2, 2g\right)^T dz\right]. \tag{18}$$

The classical Gauss-Codazzi integrability condition on the surface geometry (Appendix D) identifies (twice) the Gaussian curvature (17) with the intrinsically defined Ricci scalar given by (B.4). Thus $|\mathcal{A}|$ is completely determined once $\mathcal{V}$ and $|\psi|$ are known; it is a measure of the difference of the principal curvatures:

$$|\mathcal{A}|^2 = \frac{(C_1 - C_2)^2}{16}|\psi|^8\,. \tag{19}$$

The phase of $\mathcal{A}$ captures the principal directions; $\mathcal{A}$ vanishes only at umbilical points.

2.3 Spinor presentation of bending energy

Bending energy is quadratic in curvature. For a two-dimensional surface this implies that it must also be scale invariant. There are several different possibilities. The simplest of these is positive definite, proportional to

$$H_1 = \frac{1}{2}\int dA\, K^{ab}K_{ab} = \frac{1}{2}\int dA\,\left({C_1}^2 + {C_2}^2\right), \tag{20}$$

which vanishes on a planar region with $C_1 = 0 = C_2$ ($K_{ab} = 0$). The Canham-Helfrich energy [14]

$$H_2 = \frac{1}{2}\int dA\, K^2 = 4i\int dz\wedge d\bar{z}\,\mathcal{V}^2\,, \tag{21}$$

which vanishes on a minimal surface, assumes a remarkably simple form in the spinor representation. The conformally invariant Willmore energy is given by the integrated squared difference of the principal curvatures [28]

$$H_3 = \int dA\,\tilde{K}^{ab}\tilde{K}_{ab} = \frac{1}{2}\int dA\,(C_1 - C_2)^2\,, \tag{22}$$

where $\tilde{K}_{ab} = K_{ab} - \frac{1}{2}g_{ab}K$ is the trace-free curvature. H_3 vanishes on a sphere. It is simple to see that

$$H_3 = 4i\int dz\wedge d\bar{z}\frac{|\mathcal{A}|^2}{|\psi|^4}\,, \tag{23}$$

involving the modulus of the Wronskian $\mathcal{A}$.

Finally, one has the Gaussian bending energy, $H_4 = \int dA\, K_G$ which is identified as the Gauss-Bonnet topological invariant. It is clear that any quadratic bending energies can be constructed using any two of H_1, H_2, H_3 and H_4. Choosing one of these as H_4, it is clear that any one of the remaining three is equivalent to any other modulo a boundary term associated with the Gauss-Bonnet invariant. In particular, the Willmore energy differs from the Canham-Helfrich bending energy by a topological term linear in the Gaussian curvature. Thus, despite the functional differences between the two, their Euler-Lagrange equations had better be identical. As a consistency check, we will confirm this fact explicitly in the spinor framework in Appendix F.

3 The construction of the stress tensor

We are interested in examining the variational properties of various physically relevant energy functionals of the surface geometry within the spinor framework. This may be the area, bending energy or some other geometrical invariant. Such an invariant will generally be some scalar density, L, constructed using the spinor and the real potential $\mathcal{V}$, integrated over the the surface [2]

$$H = i \int dz \wedge d\bar{z}\, L(\psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2, \mathcal{V})\,. \tag{24}$$

Derivatives may also appear in L. Derivatives of ψ, if they occur, are expressible in terms of ψ itself and derivatives of $\ln|\psi|^2$, $\mathcal{V}$, and $\mathcal{A}$. Where higher derivatives are involved it is not entirely straightforward in this framework to identify scalar combinations of the variables appearing in the argument of L; it is facilitated, however, by taking into account the transformation properties of the densities ψ, $\mathcal{V}$ given by Eq.(11) and of $\mathcal{A}$ given by Eq.(15), and forming products with weight zero.

There are two sets of constraints that need to be accommodated when H is varied with respect to these variables: the Dirac equation (1) implies that the variations in the spinor and the potential are not independent; we also must encode how to reconstruct the surface in terms of the spinors. These constraints are enforced in the variational principle by introducing a number of Lagrange multipliers. Thus we construct the following functional:

$$\begin{aligned} H_c = H &+ i \int dz \wedge d\bar{z}\, \lambda^\dagger \mathcal{D}\psi + i \int dz \wedge d\bar{z}\, \lambda^T \mathcal{D}\xi \\ &+ i \int dz \wedge d\bar{z}\, \mathbf{f}^z \cdot (\mathbf{e}_z - \partial_z \mathbf{X}) + i \int dz \wedge d\bar{z}\, \mathbf{f}^{\bar{z}} \cdot (\mathbf{e}_{\bar{z}} - \partial_{\bar{z}} \mathbf{X})\ , \end{aligned} \tag{25}$$

where $\mathcal{D}$ is defined by Eq.(2) and $\mathbf{e}_z$ by Eq.(8). The Lagrange multipliers λ and $\mathbf{f}^z$ enforce respectively the Dirac equation and the identification of the tangent vectors[3]. It is now legitimate to treat $\mathbf{X}$, ψ, $\bar{\psi}$ and $\mathcal{V}$ as independent variables. The multiplier λ is a 2-component spinor defined by $\lambda = \left(\lambda^1, \lambda^2\right)^T$ [4]. The multipliers $\mathbf{f}^z$ form a vector in $\mathbb{E}^3$ defined by $\mathbf{f}^z = \left(f^1, f^2, f^3\right)^T$. This construction is a spinor counterpart of that for parameterized surfaces [15]. There are, however, a number of important differences in its implementation.

It may appear, at first, that the tangency constraints are redundant is the spinor framework. After all, any spinor satisfying the Dirac equation, will define a surface, and a naive counting of degrees of freedom suggests that nothing is amiss if they are dropped. This would certainly simplify the variational principle; unfortunately, it would also be wrong. The multipliers $\mathbf{f}^z$, as we will see, will get identified with the stress tensor in the surface. Failure to enforce the constraint would leads one to the invalid conclusion that equilibrium solutions always have vanishing stress which is clearly not the case.

In components, the functional H_c assumes the form[5]

$$\begin{aligned} H_c = H &+ i \int dz \wedge d\bar{z}\, \bar{\lambda}^1 \left(\partial_z \psi_2 + \mathcal{V}\psi_1\right) + i \int dz \wedge d\bar{z}\, \bar{\lambda}^2 \left(\partial_{\bar{z}}\, \psi_1 - \mathcal{V}\psi_2\right) + c.c. \\ &+ i \int dz \wedge d\bar{z} \left(\frac{f^1}{2} \left(\bar{\psi}_2{}^2 - \psi_1{}^2\right) + i \frac{f^2}{2} \left(\bar{\psi}_2{}^2 + \psi_1{}^2\right) + f^3 \psi_1 \bar{\psi}_2 - \mathbf{f}^z \cdot \partial_z \mathbf{X} \right) + c.c.\,. \end{aligned} \tag{26}$$

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[2] It will be convenient to work with a scalar density L, rather than a scalar $\mathcal{L}$; the two are related by $L = \frac{1}{2}|\psi|^4 \mathcal{L}$.

[3] As defined, both λ and $\mathbf{f}^z$ are densities.

[4] The fact that that the Dirac equation (4) is the complex conjugate of (1) is reflected in the fact that the multipliers enforcing these conditions are also complex conjugates. In addition, $\mathbf{f}^{\bar{z}} = \bar{\mathbf{f}}^z$.

[5] c.c. represents the complex conjugate expression.

Begin with the variation of the embedding functions $\mathbf{X}$. Performing integration by parts to collect the derivatives of $\mathbf{f}^z$ and $\mathbf{f}^{\bar{z}}$ in a divergence, one finds

$$\delta_{\mathbf{X}} H_c = i \int dz \wedge d\bar{z} \left(\partial_z \mathbf{f}^z \cdot \delta \mathbf{X} - \partial_z (\mathbf{f}^z \cdot \delta \mathbf{X}) \right) + c.c. \, .$$

Thus, the Euler-Lagrange derivative with respect to $\mathbf{X}$, $\varepsilon_{\mathbf{X}} \equiv \delta H_c / \delta \mathbf{X}$ is identified as

$$\varepsilon_{\mathbf{X}} = \partial_z \mathbf{f}^z + \partial_{\bar{z}} \mathbf{f}^{\bar{z}} \, . \tag{27}$$

Critical points of H_c satisfy the Euler-Lagrange equations, $\varepsilon_{\mathbf{X}} = 0$. The solutions of the Euler-Lagrange equations are therefore described in terms of the conserved complex-valued surface "stress tensor", $\mathbf{f}^z$ [6]. It remains to construct $\mathbf{f}^z$ explicitly. This will involve the solution of the Euler-Lagrange equations for the variables ψ, $\bar{\psi}$ and $\mathcal{V}$.

Varying H_C with respect to the potential $\mathcal{V}$ gives

$$\delta_{\mathcal{V}} H_C = i \int dz \wedge d\bar{z} \left(\frac{\delta L}{\delta \mathcal{V}} + \bar{\lambda}^1 \psi_1 + \lambda^1 \bar{\psi}_1 - \bar{\lambda}^2 \psi_2 - \lambda^2 \bar{\psi}_2 \right) \delta \mathcal{V} \, ;$$

the corresponding Euler-Lagrange equation is thus

$$\varepsilon_{\mathcal{V}} = \lambda^\dagger \sigma_3 \psi + \lambda^T \sigma_3 \bar{\psi} + \frac{\delta L}{\delta \mathcal{V}} = 0 \, . \tag{28}$$

Here σ_3 is the Pauli matrix with 1 and -1 along the diagonal. Since $\mathcal{V}$ is real, this equation is also real. The spinor λ thus satisfies a single inhomogeneous linear algebraic equation in a two-dimensional complex vector space. One solution of this equation is given by

$$\lambda_i = -\frac{1}{2|\psi|^2} \frac{\delta L}{\delta \mathcal{V}} \sigma_3 \psi \, . \tag{29}$$

However, this solution is not unique. Let us write Eq.(28) in the form $Re(\lambda^\dagger \sigma_3 \psi) = -\delta L / \delta \mathcal{V}$. Any spinor λ with an imaginary projection onto $\sigma_3 \psi$ is evidently a solution of the homogeneous equation $Re(\lambda^\dagger \sigma_3 \psi) = 0$. This equation has two solutions. The first of these is of the form

$$\lambda_h^r = i r \sigma_3 \psi, \tag{30}$$

where r is an arbitrary real-valued function. This is because $\lambda_h^{r\,\dagger} \sigma_3 \psi = -i |\psi|^2 r$ which is manifestly imaginary. The second solution, which involves the spinor ξ, is given by

$$\lambda_h^c = \bar{c} \sigma_3 \xi, \tag{31}$$

where c is an arbitrary complex-valued function. This solution satisfies $\lambda_h^\dagger \sigma_3 \psi = 0$ on account of the orthogonality of ψ and ξ.

Thus the complete solution $\lambda = \lambda_i + \lambda_h$ is given by

$$\lambda = \left(-\frac{1}{2|\psi|^2} \frac{\delta L}{\delta \mathcal{V}} + i r \right) \sigma_3 \psi + \bar{c} \sigma_3 \xi \, , \tag{32}$$

the components of which read

$$\lambda^1 = \left(-\frac{1}{2|\psi|^2} \frac{\delta L}{\delta \mathcal{V}} + i r \right) \psi_1 - \bar{c} \bar{\psi}_2, \qquad \lambda^2 = \left(\frac{1}{2|\psi|^2} \frac{\delta L}{\delta \mathcal{V}} - i r \right) \psi_2 - \bar{c} \bar{\psi}_1 \, . \tag{33}$$

This solution possesses 3 degrees of freedom per point: one for r and two for c. To justify the counting, note that in terms of real-valued variables, the solution of equation (28) describes a 3-dimensional hyperplane in a 4-dimensional vector space equipped with a (non-degenerate) inner product of signature $(+,+,-,-)$. It will be shown below that the ambiguity reflected in the functions r and c is a gauge artifact associated with the parametrization.

[6] With a densitized stress tensor covariant differentiation is replaced by partial differentiation.

Let us introduce the quantities $f^+ = f^1 + if^2$ and $f^- = f^1 - if^2$. The Euler-Lagrange equations for the spinor ψ are given by[7]

$$\begin{aligned} \varepsilon_{\psi_1} &= -\psi_1 f^- + \bar{\psi}_2 f^3 + T^1 = 0\,, && (34a)\\ \varepsilon_{\psi_2} &= \psi_2 \bar{f}^+ + \bar{\psi}_1 \bar{f}^3 + T^2 = 0\,, && (34b)\end{aligned}$$

where

$$T^1 = \frac{\delta L}{\delta \psi_1} - \partial_{\bar{z}}\bar{\lambda}^2 + \bar{\lambda}^1 \mathcal{V}\,, \qquad T^2 = \frac{\delta L}{\delta \psi_2} - \partial_z \bar{\lambda}^1 - \bar{\lambda}^2 \mathcal{V}\,. \qquad (35)$$

together with their complex conjugate counterparts.

Solving this set of equations for $\mathbf{f}^z$ is facilitated by by first expressing the Cartesian components f^+, f^- and f^3 in terms of their geometrically more relevant counterparts with respect to the basis of tangent vectors adapted to the surface $\{\mathbf{e}_z, \mathbf{e}_{\bar{z}}, \mathbf{n}\}$. The latter decomposition of $\mathbf{f}^z$ is given by

$$\sqrt{g}\,\mathbf{f}^z = i\left(\mathbf{f}^z\cdot\mathbf{e}_{\bar{z}}\,\mathbf{e}_z + \mathbf{f}^z\cdot\mathbf{e}_z\,\mathbf{e}_{\bar{z}}\right) + \sqrt{g}\,\mathbf{f}^z\cdot\mathbf{n}\,\mathbf{n}\,. \qquad (36)$$

The three projections can be expressed in terms of the Cartesian components, f^+, f^-, f^3, as follows

$$\begin{aligned} f^z{}_z &= \mathbf{f}^z\cdot\mathbf{e}_z = \frac{1}{2}\left(\bar{\psi}_2{}^2 f^+ - \psi_1{}^2 f^-\right) + \psi_1\bar{\psi}_2 f^3, && (37a)\\ f^z{}_{\bar{z}} &= \mathbf{f}^z\cdot\mathbf{e}_{\bar{z}} = -\frac{1}{2}\left(\bar{\psi}_1{}^2 f^+ - \psi_2{}^2 f^-\right) + \bar{\psi}_1\psi_2 f^3, && (37b)\\ f^z &= \mathbf{f}^z\cdot\mathbf{n} = \frac{1}{|\psi|^2}\left(\bar{\psi}_1\bar{\psi}_2 f^+ + \psi_1\psi_2 f^- + (|\psi_1|^2 - |\psi_2|^2) f^3\right)\;; && (37c)\end{aligned}$$

these three equations are now inverted in favor of f^+, f^- and f^3 to obtain

$$\begin{aligned} f^+ &= \frac{2}{|\psi|^4}\left(\psi_2{}^2 f^z{}_z - \psi_1{}^2 f^z{}_{\bar{z}} + |\psi|^2\psi_1\psi_2 f^z\right), && (38a)\\ f^- &= \frac{2}{|\psi|^4}\left(-\bar{\psi}_1{}^2 f^z{}_z + \bar{\psi}_2{}^2 f^z{}_{\bar{z}} + |\psi|^2\bar{\psi}_1\bar{\psi}_2 f^z\right), && (38b)\\ f^3 &= \frac{2}{|\psi|^4}\left(\bar{\psi}_1\psi_2 f^z{}_z + \psi_1\bar{\psi}_2 f^z{}_{\bar{z}}\right) + \frac{1}{|\psi|^2}\left(|\psi_1|^2 - |\psi_2|^2\right) f^z\,. && (38c)\end{aligned}$$

Substituting these expressions into the EL equations (34a) and (34b) one obtains

$$\begin{aligned} \varepsilon_{\psi_1} &= \frac{2}{|\psi|^2}\bar{\psi}_1 f^z{}_z - \bar{\psi}_2 f^z + T^1 = 0, && (39)\\ \varepsilon_{\bar{\psi}_2} &= \frac{2}{|\psi|^2}\psi_2 f^z{}_z + \psi_1 f^z + \bar{T}^2 = 0\,. && (40)\end{aligned}$$

In particular, the EL equations for ψ_1 and $\bar{\psi}_2$ (and the conjugates of these equations) involve $f^z{}_z$ and f^z (and their complex conjugates) but not $f^z{}_{\bar{z}}$ (or its complex conjugate). The tangential projection $f^z{}_{\bar{z}}$ remains undetermined at this level. At an algebraic level, this fact is related to the identity $\mathbf{e}_z = i\mathbf{e}_z \times \mathbf{n}$.

The combination $\psi_1\,\varepsilon_{\psi_1} + \bar{\psi}_2\,\varepsilon_{\bar{\psi}_2}$ determines $f^z{}_z$:

$$\psi_1\,\varepsilon_{\psi_1} + \bar{\psi}_2\,\varepsilon_{\bar{\psi}_2} = 2f^z{}_z + \psi_1 T^1 + \bar{\psi}_2\bar{T}^2 = 0\,. \qquad (41)$$

Using expressions (35) for T^1 and T^2 in this equation and solving for $f^z{}_z$ one finds

$$f^z{}_z = -\frac{1}{2}\left(\psi_1\frac{\delta L}{\delta\psi_1} + \bar{\psi}_2\frac{\delta L}{\delta\bar{\psi}_2}\right) + \frac{1}{2}\partial_{\bar{z}}\left(\lambda^1\bar{\psi}_2 + \bar{\lambda}^2\psi_1\right) - i\mathcal{V}\,Im(\lambda^\dagger\psi)\,.$$

Now, by substituting into this last equation the expressions for λ^1 and λ^2 given in (33) one finally obtains

$$f^z{}_z = -\frac{1}{2}\left(\psi_1\frac{\delta L}{\delta\psi_1} + \bar{\psi}_2\frac{\delta L}{\delta\bar{\psi}_2}\right) + i\psi_1\bar{\psi}_2\partial_{\bar{z}} r - \frac{1}{2}\left(\bar{\psi}_2{}^2\partial_{\bar{z}}\bar{c} + \psi_1{}^2\partial_{\bar{z}} c\right)\,. \qquad (42)$$

[7] Components of the complex conjugate of the spinor, $\bar{\psi}_1$ and $\bar{\psi}_2$, are varied independently of ψ_1 and ψ_2.

Similarly f^z is determined by the combination $\bar\psi_1\,\varepsilon_{\bar\psi_2} - \psi_2\,\varepsilon_{\psi_1}$:

$$\bar\psi_1\,\varepsilon_{\bar\psi_2} - \psi_2\,\varepsilon_{\psi_1} \;=\; |\psi|^2 f^z + \bar\psi_1 \bar T^2 - \psi_2 T^1 = 0. \tag{43}$$

Substitution of expressions for T^1 and T^2 into this equation and solving for f^z gives

$$\begin{aligned} f^z &= \frac{1}{|\psi|^2}\left(\psi_2 \frac{\delta L}{\delta \psi_1} - \bar\psi_1 \frac{\delta L}{\delta \bar\psi_2} + \partial_{\bar z}\left(\lambda^1\bar\psi_1 - \bar\lambda^2\psi_2\right)\right) \\ &+ \frac{1}{|\psi|^2}\left(\bar\lambda^2 \partial_{\bar z}\psi_2 - \lambda^1\partial_{\bar z}\bar\psi_1 + \mathcal{V}(\bar\lambda^1\psi_2 + \lambda^2\bar\psi_1)\right)\,. \end{aligned}$$

Using once again expressions (33) in place of λ^1 and λ^2 along with the the expressions (C.2) and (C.3) for the derivatives of ψ_1 and ψ_2, the following simplification results

$$\begin{aligned} f^z &= -\partial_{\bar z}\left(\frac{1}{2|\psi|^2}\frac{\delta L}{\delta \mathcal{V}}\right) + \frac{1}{|\psi|^2}\left(\psi_2 \frac{\delta L}{\delta\psi_1} - \bar\psi_1\frac{\delta L}{\delta\bar\psi_2}\right) \\ &+ \frac{i}{|\psi|^2}(|\psi_1|^2 - |\psi_2|^2)\partial_{\bar z} r + \frac{1}{|\psi|^2}\left(\psi_1\psi_2\partial_{\bar z}c - \bar\psi_1\bar\psi_2\partial_{\bar z}\bar c\right)\,. \end{aligned} \tag{44}$$

The only ambiguity in the components ${f^z}_z$ and f^z of the stress tensor is the one inherited from the solution of Eq.(28).

As noted previously, the Euler-Lagrange equations for the spinor and the potential leave completely undetermined the off-diagonal component ${f^z}_{\bar z}$ of the tangential stress. At first, this appears to suggest that something is amiss. One must remember, however, that by representing the surface isothermally, one necessarily foregoes access to reparametrization invariance. This feature manifests itself in the tangential projections of the conservation law for the stress tensor. Whereas these equations would be satisfied identically in any completely reparametrization invariant framework, in this one they provide the differential equations determining the missing component of the tangential stress.

Taking the projections of the conservation law $\varepsilon_{\mathbf{X}} = 0$, where $\varepsilon_{\mathbf{X}}$ is given by Eq.(27), onto the tangent vectors provide the equation

$$\varepsilon_{\mathbf{X}}\cdot\mathbf{e}_z = \partial_{\bar z}{f^{\bar z}}_z + |\psi|^4\partial_z\left(\frac{1}{|\psi|^4}{f^z}_z\right) + \mathcal{A}\,f^z + |\psi|^2\mathcal{V}f^{\bar z} = 0, \tag{45}$$

along with its complex conjugate expression. Thus far, it has not been necessary to specify explicitly the functional form of L. To solve Eq.(45), we will suppose for simplicity that L depends only on $|\psi|^2$ and $\mathcal{V}$, undifferentiated. These differential equations can then be solved for the missing component of the stress tensor. The most general solution is given by

$${f^z}_{\bar z} = \frac{\bar{\mathcal{A}}}{2\,|\psi|^2}\frac{\delta L}{\delta\mathcal{V}} + i\bar\psi_1\psi_2\partial_{\bar z} r + \frac{1}{2}\left({\psi_2}^2\partial_{\bar z}c + {\bar\psi_1}^2\partial_{\bar z}\bar c\right) + \bar h(\bar z)\,, \tag{46}$$

where $h(z)$ is an arbitrary function.

The projection onto the normal vector provides the "shape" equation

$$\varepsilon_{\mathbf{X}}\cdot\mathbf{n} = \partial_z f^z + \partial_{\bar z}f^{\bar z} - \frac{K}{2}\left({f^z}_z + {f^{\bar z}}_{\bar z}\right) - \frac{2}{|\psi|^4}\left(\mathcal{A}{f^z}_{\bar z} + \bar{\mathcal{A}}{f^{\bar z}}_z\right) = 0\,. \tag{47}$$

We are now in a position to examine the different ambiguities which have arisen in our construction of the stress tensor. The first of these originates in the solution of the Euler-Lagrange equation for the potential.

The contribution to the stress originating in the homogeneous solution λ^c_h is given by

$$\begin{aligned} |\psi|^4\mathbf{f}^z_c &= \partial_{\bar z}c\left({\psi_2}^2\mathbf{e}_z - {\psi_1}^2\mathbf{e}_{\bar z} + |\psi|^2\psi_1\psi_2\mathbf{n}\right) \\ &+ \partial_{\bar z}\bar c\left({\bar\psi_1}^2\mathbf{e}_z - {\bar\psi_2}^2\mathbf{e}_{\bar z} - |\psi|^2\bar\psi_1\bar\psi_2\mathbf{n}\right)\,. \end{aligned} \tag{48}$$

Decomposing the complex-valued function c into its real and imaginary parts, $c = c_x + ic_y$, it is possible to express this contribution as the partial derivative of a space vector

$$\mathbf{f}^z_c = i\partial_{\bar z}\left(c_y\hat{\mathbf{x}}_1 + c_x\hat{\mathbf{x}}_2\right) , \tag{49}$$

where the fact that the basis vectors of $\mathbb{E}^3$ are constant has been used (holomorphic will do). With respect to the surface adopted basis these three vectors are given by

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \frac{1}{|\psi|^4}\left(({\psi_2}^2 - {\bar\psi_1}^2)\mathbf{e}_z + ({\bar\psi_2}^2 - {\psi_1}^2)\mathbf{e}_{\bar z} + |\psi|^2(\psi_1\psi_2 + \bar\psi_1\bar\psi_2)\mathbf{n}\right) , && \text{(50a)}\\
\hat{\mathbf{x}}_2 &= \frac{i}{|\psi|^4}\left(-({\psi_2}^2 + {\bar\psi_1}^2)\mathbf{e}_z + ({\bar\psi_2}^2 + {\psi_1}^2)\mathbf{e}_{\bar z} - |\psi|^2(\psi_1\psi_2 - \bar\psi_1\bar\psi_2)\mathbf{n}\right) , && \text{(50b)}\\
\hat{\mathbf{x}}_3 &= \frac{2}{|\psi|^4}\left(\bar\psi_1\psi_2\mathbf{e}_z + \psi_1\bar\psi_2\mathbf{e}_{\bar z} + \frac{1}{2}|\psi|^2(|\psi_1|^2 - |\psi_2|^2)\mathbf{n}\right) . && \text{(50c)}
\end{aligned}$$

Similarly, the contribution to the stress tensor originating in λ^r_h is given by

$$\mathbf{f}^z_r = i\partial_{\bar z}\left(r\hat{\mathbf{x}}_3\right) , \tag{51}$$

so that in full the contribution to the stress tensor arising from the homogeneous solution can be expressed as the partial derivative of a real-valued space vector $\mathbf{V}$,

$$\mathbf{f}^z_h = i\partial_{\bar z}\mathbf{V} , \tag{52}$$

where $\mathbf{V}$ is given by $\mathbf{V} = (c_y\hat{\mathbf{x}}_1 + c_x\hat{\mathbf{x}}_2 + r\hat{\mathbf{x}}_3)$. Since $\mathbf{V}$ is real-valued, $\mathbf{f}^z_h$ has zero divergence: $\partial_z\mathbf{f}^z_h + \partial_{\bar z}\mathbf{f}^{\bar z}_h = 0$. It is a null tensor which is automatically conserved and does not contribute to the shape equation (47). Therefore, as claimed above, it is legitimate to neglect this contribution to the stress tensor and retain only the part arising from the inhomogeneous solution λ_i.

Unlike the canonical stress tensor in the parametrized description of a surface, its counterpart in this framework is not unique.

The ambiguity associated with the arbitrary function $h(z)$ appearing in the solution of Eq.(45) given by Eq.(46) appears to be of a more serious nature. It is not simply a gauge artifact, contributing as it does to the shape equation a term

$$\approx \frac{2}{|\psi|^4}\left(\mathcal{A}\bar h(\bar z) + \bar{\mathcal{A}}h(z)\right) .$$

However, the normal projection of the Euler-Lagrange derivative $\varepsilon_{\mathbf{X}}\cdot\mathbf{n}$ associated with any reparametrization invariant energy function had better form a scalar (density). Under a holomorphic transformation $z \to w(z)$ the factor $\mathcal{A}/|\psi|^4$ transforms as

$$\frac{\mathcal{A}(z,\bar z)}{|\psi(z,\bar z)|^4} \to \frac{w'(z)}{\bar w'(\bar z)}\frac{\mathcal{A}(w,\bar w)}{|\psi(w,\bar w)|^4} . \tag{53}$$

In order to form a scalar $h(z)$ should transform as $h(z) \to w'(z)/\bar w'(\bar z)h(w)$, which is not a holomorphic function. Consistency requires that h must vanish.

In summary, for a functional which depends on $|\psi|$ and $\mathcal{V}$, the components of the stress tensor assume the simple form

$$\begin{aligned}
{f^z}_z &= -\frac{1}{2}\left(\psi_1\frac{\partial L}{\partial\psi_1} + \bar\psi_2\frac{\partial L}{\partial\bar\psi_2}\right) , && \text{(54a)}\\
{f^z}_{\bar z} &= \frac{\bar{\mathcal{A}}}{2\,|\psi|^2}\frac{\partial L}{\partial\mathcal{V}} , && \text{(54b)}\\
f^z &= -\partial_{\bar z}\left(\frac{1}{2|\psi|^2}\frac{\partial L}{\partial\mathcal{V}}\right) . && \text{(54c)}
\end{aligned}$$

The Euclidean invariance of the energy and the identification of $\mathbf{f}^z$ with the stress tensor within this framework is discussed in Appendix E. A more general dependence on the spinor and potential is considered in Appendix F.

3.1 Canham-Helfrich energy

In soft matter applications, the surface energy will typically be a sum of bending energy, a term linear in K reflecting an asymmetry between the two sides of the surface, and an area term associated with a constraint or penalty on the total area [13],

$$L = \frac{1}{2}\kappa L_2 + \beta L_1 + \sigma L_0 \,, \tag{55}$$

where $L_n = 1/2|\psi|^4 K^n$.

One finds that
$$\frac{\partial L_n}{\partial \psi_i} = (1 - \frac{n}{2})|\psi|^2 K^n \bar{\psi}_i \qquad \text{and} \qquad \frac{\partial L_n}{\partial \mathcal{V}} = 2\; n\,|\psi|^2 K^{n-1} \,. \tag{56}$$

From equations (54) one reads off the contribution of L_n to the various components of the stress tensor:

$$\begin{aligned} {f_n}^z{}_z &= \frac{(n-2)}{4}|\psi|^4 K^n \,, && (57a)\\ {f_n}^z{}_{\bar{z}} &= n\,\bar{\mathcal{A}} K^{n-1} \,, && (57b)\\ {f_n}^z &= -\,n\,\partial_{\bar{z}} K^{n-1} \,. && (57c) \end{aligned}$$

It is simple to confirm that these expressions reproduce the well-known result [16],

$$\mathbf{f}^a_n = K^{n-1}(nK^{ab} - Kg^{ab})\,\mathbf{e}_b - n\nabla^a K^{n-1}\,\mathbf{n} \,, \tag{58}$$

in this particular parametrization. The corresponding contribution to the normal component of the Euler-Lagrange derivative (47) is also easily shown to be given by

$$\mathcal{E}_n = -n\Delta K^{n-1} + K^{n-1}\left(2nK_G - (n-1)K^2\right) \,. \tag{59}$$

n=0: Area

For surface area, corresponding to $L_0 = 1/2|\psi|^4$, only the diagonal tangential stress is non-vanishing:

$$f^z{}_z = -\frac{1}{2}|\psi|^4, \quad f^z{}_{\bar{z}} = 0, \quad f^z = 0 \,. \tag{60}$$

The Euler-Lagrange derivative is given by $\mathcal{E}_0 = K$; the critical points of area are minimal surfaces satisfying $K = 0$ or $\mathcal{V} = 0$.

n=1: Integrated mean curvature

For an energy proportional to the mean curvature, corresponding to $L_1 = 1/2|\psi|^4 K$, the components of the tangential stress tensor are

$$f^z{}_z = -\frac{1}{4}|\psi|^4 K, \quad f^z{}_{\bar{z}} = \bar{\mathcal{A}} \,, \tag{61}$$

and the normal stress vanishes, $f^z = 0$. The Euler-Lagrange derivative $\mathcal{E}_1 = \mathcal{R}$; the critical points are developable with vanishing Gauss curvature.

n=2: Canham-Helfrich bending energy $L_B = L_2/2 = 4\mathcal{V}^2$

The components of the stress tensor are

$$f^z{}_z = 0, \quad f^z{}_{\bar{z}} = \bar{\mathcal{A}} K, \quad f^z = -\partial_{\bar{z}} K \,, \tag{62}$$

and the Euler-Lagrange derivative given by

$$\mathcal{E}_B = -\Delta K + K\left(2K_G - \frac{1}{2}K^2\right) \,. \tag{63}$$

The vanishing of $f^z{}_z$ in this case is a manifestation of scale invariance.

It was pointed out in the paragraph following Eq.(25) that if the constraints–defining the stress tensor–relating the tangents to the embedding variables are not enforced, the Euler-Lagrange equations obtained

are incorrect. To underscore this point, consider the consequence of dropping this constraint so that $\mathbf{f}^a = 0$; the surface states are stress-free. In the case of pure bending, setting $\mathbf{f}^a = 0$ in Eq.(58) for $n = 2$ implies that $K = 0$ or $K_{ab} = g_{ab}K/2$; the latter possibility implies a spherical geometry. The only stress-free solutions are thus minimal surfaces or spheres. Even though the variations of ψ and $\mathcal{V}$ are consistent with the Dirac equation, and thus represent a surface, without the additional constraints, the corresponding Euler-Lagrange equations do not describe the critical points of the surface problem correctly.

4 Discussion

We have provided a variational framework, tailored to the representation of the surface geometry in terms of a spinor field interacting through a potential, to describe the equilibrium properties of this surface. The construction of a conserved stress tensor lies at the center of this framework. Our primary aim, of course, was not to provide yet another construction of the stress tensor. Even if this derivation does provide significant new insight into its relationship with the surface geometry, there are other derivations; rather it has been to examine how these variables can be reconciled with surface variational principles. One is now in a position to import the techniques of complex analysis to re-examine various inherently non-linear problems in soft matter where our toolbox has been found wanting.

A problem that lends itself to be treated using the spinor representation of the surface geometry is a very old one that has been the focus of a revival of interest in recent years: how does a surface bend when its local geometry is constrained; a good example is provided by an unstretchable planar sheet of paper. Contrary to initial expectations, this is not a simple problem [29, 30, 31]. The general configuration will consist of piecewise developable surfaces (with vanishing Gaussian curvature) meeting along a set of ridges. Such surfaces assume a particularly simple form in the spinor representation on account of Eq.(B.4). If the Gaussian curvature vanishes, $|\psi|$ itself is harmonic. Unfortunately, developable surfaces do not generally occur as solutions of the unadorned Euler-Lagrange equations for bending: the correct Euler-Lagrange equations possess an additional term arising from the local constraint on the metric that needs to be imposed to maintain flatness under variation; they do not minimize bending energy unless one imposes this additional local constraint [32]. One also finds that the boundary conditions that are appropriate will reflect the constraint on the metric. A simple but non-trivial example is provided by a cone; it does not minimize bending energy unless the metric is constrained to be flat. This is why paper folds into cones but fluid membranes do not! This local constraint is very different in nature from the structural constraints appearing in the auxiliary framework; whereas the role of the latter set of constraints is to ensure that the spinors and the potential are consistent with some spatial geometry, they do not place any constraint on the geometry itself. It is remarkable therefore that the local constraint can be treated technically on an identical footing [32]. The extra term takes the form of a linear coupling of the corresponding tangential stress to extrinsic curvature. We are currently examining this problem within the spinor framework.

There are several interesting directions for future work. Statistical field theory is an obvious one, where we have been largely limited to Gaussian approximations using traditional methods. There is, however, still some spadework to be done before one is in a position to do this with any confidence. It would be useful to first address a few questions of an elementary nature: how does one describe perturbation theory about some given equilibrium surface in this representation? Does the simple form of the bending energy have a counterpart in the expansion at second order? In this context, it is clear that even if one is interested in minimal surfaces described by the WE representation with $\mathcal{V} = 0$, one needs to introduce a potential in the variation. Finally, it would appear that the coupling of electrons to the curved surface geometry in graphene calls naturally for a spinor description of the surface.

Partial support from DGAPA PAPIIT grant IN114510 is acknowledged.

Appendix A Rotation of the surface in $\mathbb{E}^3$

The Dirac equation is not $SU(2)$ invariant. Nor are rotations of the surface realized in what would appear to be the "obvious" way by the action of $SU(2)$ on the spinor ψ using the local isomorphism $SO(3) = SU(2)$. Note, however, that the surface which corresponds to the spinor ξ is the same as that obtained from ψ under a rotation by an angle π around the $\hat{\mathbf{x}}_2$ axis, i.e. under the substitution $\psi \to \xi$ the components of $\mathbf{X}$ change as $X^1 \to -X^1, X^2 \to X^2, X^3 \to -X^3$. More generally, the transformed spinor

$\tilde{\psi} = \bar{a}\psi \;+ b\,\xi$, where a and b are two complex numbers, also satisfies the Dirac equation so that it, as well, describes a surface $\tilde{\mathbf{X}}$. If $|a|^2 + |b|^2 = 1$, $\tilde{\psi}$ is indirectly related to ψ by an element U of $SU(2)$ given by

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \tag{A.1}$$

which can also be represented as $U = e^{-i\frac{\vartheta}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}}$, where $\hat{\mathbf{r}}$ is a unit vector in $\mathbb{E}^3$ and $\boldsymbol{\sigma}$ is the vector in $\mathbb{E}^3$ with the Pauli matrices as its components. Because of the local isomorphism $SO(3) = SU(2)$ this identifies the element R of $SO(3)$ given by $\mathsf{R} = e^{i\vartheta\mathbf{J}\cdot\hat{\mathbf{r}}}$, where $\mathbf{J}$ is a vector in $\mathbb{E}^3$ with the infinitesimal generators of rotations as its components, so that R describes a anticlockwise rotation by an angle ϑ around the direction $\hat{\mathbf{r}}$. Evidently $\tilde{\psi} \neq U\psi$, thus the manner in which U relates $\tilde{\psi}$ and ψ is not through the usual action of $SU(2)$, but with its associated rotation R that relates the embedding functions of both spinors, namely the embedding functions $\tilde{\mathbf{X}}$ describe a rotation of $\mathbf{X}$, i.e. $\tilde{\mathbf{X}} = \mathsf{R}\mathbf{X}$ [12]. If $|a|^2 + |b|^2 \neq 1$, the transformation $\psi \to \bar{a}\psi \;+ b\,\xi$ describes a rotation accompanied by a scaling.

Appendix B The Laplace-Beltrami operator and the Ricci scalar

The Laplace-Beltrami operator Δ on the complex plane is given by

$$\Delta = \frac{4}{|\psi|^4}\partial_z\partial_{\bar{z}}\,. \tag{B.1}$$

The non-vanishing Christoffel symbols constructed with the metric g_{ab} are

$$\Gamma^z{}_{zz} = g^{z\bar{z}}\partial_z g_{z\bar{z}} = \partial_z \ln g_{z\bar{z}} = 2\,\partial_z \ln|\psi|^2\,, \tag{B.2}$$

and its complex conjugate $\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}}$. Likewise, the only non-vanishing components of the Riemann tensor in the conformal parametrization are

$$R^z{}_{z\bar{z}z} = \frac{|\psi|^4}{2}\Delta\ln|\psi|^2 = R^{\bar{z}}{}_{\bar{z}z\bar{z}}\,. \tag{B.3}$$

The Ricci tensor (proportional to the metric) and the Ricci scalar are

$$R_{z\bar{z}} = -\frac{|\psi|^4}{2}\Delta\ln|\psi|^2, \qquad \mathcal{R} = -2\Delta\ln|\psi|^2\,. \tag{B.4}$$

Appendix C Identities for $\partial_z\psi_1$ and $\partial_z\bar{\psi}_2$

The two symmetric curvature scalars depend on ψ only through the combinations $|\psi|^2$ and $\mathcal{A}$. It is useful to possess identities for the missing partial derivatives, $\partial_z\psi_1$ and $\partial_z\bar{\psi}_2$, in terms of these variables.

Begin by differentiating $|\psi|^2$ with respect to z, using the fact that ψ satisfies the Dirac equation, to obtain

$$\partial_z|\psi|^2 = \psi_2\partial_z\bar{\psi}_2 + \bar{\psi}_1\partial_z\psi_1\,. \tag{C.1}$$

Now multiply across by ψ_1 and add $\psi_2\mathcal{A}$ to both sides to get

$$\psi_2\mathcal{A} + \psi_1\partial_z|\psi|^2 = |\psi|^2\partial_z\psi_1\,,$$

we thus obtain the identity

$$\partial_z\psi_1 = \psi_1\partial_z\ln|\psi|^2 + \frac{1}{|\psi|^2}\psi_2\mathcal{A}\,. \tag{C.2}$$

Multiplying Eq.(C.1) across by $\bar{\psi}_2$ and subtracting $\bar{\psi}_1\mathcal{A}$ from both sides yields

$$\bar{\psi}_2\partial_z|\psi|^2 - \bar{\psi}_1\mathcal{A} = |\psi|^2\partial_z\bar{\psi}_2\,,$$

giving the second identity

$$\partial_z\bar{\psi}_2 = \bar{\psi}_2\partial_z\ln|\psi|^2 - \frac{1}{|\psi|^2}\bar{\psi}_1\mathcal{A}\,. \tag{C.3}$$

All derivatives of ψ can now be expressed in the compact form,

$$\partial_z\psi = \begin{pmatrix} \partial_z\ln|\psi|^2 & |\psi|^{-2}\mathcal{A} \\ -\mathcal{V} & 0 \end{pmatrix}\psi\,, \qquad \partial_{\bar{z}}\psi = \begin{pmatrix} 0 & \mathcal{V} \\ -|\psi|^{-2}\bar{\mathcal{A}} & \partial_{\bar{z}}\ln|\psi|^2 \end{pmatrix}\psi\,. \tag{C.4}$$

Appendix D Gauss-Weingarten equations and the integrability conditions

The Gauss-Weingarten equations, $\partial_a \mathbf{e}_a = \Gamma^c{}_{ab}\mathbf{e}_c - K_{ab}\mathbf{n}$ and $\partial_a \mathbf{n} = K_a{}^b\,\mathbf{e}_b$, describe how the adopted frame $\{\mathbf{e}_a, \mathbf{n}\}$ changes as it is moved across the surface. In the generalized WE representation, by writing the adapted basis to the surface as $\mathbf{E} = \{\mathbf{e}_z, \mathbf{e}_{\bar{z}}, \mathbf{n}\}$, these equations can be expressed in the compact form [11]

$$\partial_z \mathbf{E} = \mathsf{M}\,\mathbf{E} \qquad \partial_{\bar{z}}\mathbf{E} = \mathsf{N}\,\mathbf{E}, \tag{D.1}$$

where the linear transformations M and N are given by

$$\mathsf{M} = \begin{pmatrix} 2\partial_z \ln|\psi|^2 & 0 & -\mathcal{A} \\ 0 & 0 & -\frac{1}{4}K|\psi|^4 \\ \frac{1}{2}K & \frac{2}{|\psi|^4}\mathcal{A} & 0 \end{pmatrix}, \quad \mathsf{N} = \begin{pmatrix} 0 & 0 & -\frac{1}{4}K|\psi|^4 \\ 0 & 2\partial_{\bar{z}} \ln|\psi|^2 & -\bar{\mathcal{A}} \\ \frac{2}{|\psi|^4}\bar{\mathcal{A}} & \frac{1}{2}K & 0 \end{pmatrix}. \tag{D.2}$$

The integrability condition $\mathbf{E}_{z\bar{z}} = \mathbf{E}_{\bar{z}z}$ leads to $\mathsf{M}_{\bar{z}} - \mathsf{N}_z + [\mathsf{M}, \mathsf{N}] = 0$. From this condition and the linear independence of basis $\mathbf{E}$ one obtains the following relations

$$\partial_z\partial_{\bar{z}} \ln|\psi|^2 = \frac{|\mathcal{A}|^2}{|\psi|^4} - \mathcal{V}^2 = \frac{|\mathcal{A}|^2}{|\psi|^4} - \frac{|\psi|^4}{16}K^2\,, \tag{D.3a}$$

$$\partial_{\bar{z}}\mathcal{A} = |\psi|^2\partial_z\mathcal{V} - \mathcal{V}\partial_z|\psi|^2 = \frac{|\psi|^4}{4}\partial_z K\,, \tag{D.3b}$$

which are fulfilled by spinors and potential satisfying the Dirac-type Eq.(1). So it is unnecessary to implement them in the effective functional when performing the variational principle.
Equation (D.3a) is the Gauss-Codazzi equation since it is equivalent to the equation

$$\mathcal{R} = K^2 - K_{ab}K^{ab} = K^2 - (K_{zz}K^{zz} + 2K_{z\bar{z}}K^{z\bar{z}} + K_{\bar{z}\bar{z}}K^{\bar{z}\bar{z}}). \tag{D.4}$$

Likewise, equation (D.3b) and its complex conjugate are the Codazzi-Mainardi equations for they are equivalent to the pair of equations

$$\nabla_a K_{bc} = \nabla_b K_{ac} \qquad \text{or} \qquad \nabla_{\bar{z}} K_{zz} = \nabla_z K_{\bar{z}z} \quad \text{and its c.c.}\,. \tag{D.5}$$

Appendix E Conserved stress and torque

The current associated with the variation of the embedding functions, denoted by $Q_{\mathbf{X}}$, is identified with the boundary term in the variation of the functional, H_c, with respect to $\mathbf{X}$, or

$$Q_{\mathbf{X}} = -i\int dz \wedge d\bar{z}\,\left(\partial_z(\mathbf{f}^z\cdot\delta\mathbf{X}) + \partial_{\bar{z}}(\mathbf{f}^{\bar{z}}\cdot\delta\mathbf{X})\right)\,. \tag{E.1}$$

The current associated with the variations in the spinor field is given by

$$Q_{\psi_1} = i\int dz \wedge d\bar{z}\partial_{\bar{z}}(\bar{\lambda}^2\delta\psi_1), \qquad Q_{\psi_2} = i\int dz\wedge d\bar{z}\partial_z(\bar{\lambda}^1\delta\psi_2), \tag{E.2}$$

together with their complex conjugate expressions.

The potential does not give rise to a current. Thus the complete variation of H_c is given by

$$\delta H_c = i\int dz\wedge d\bar{z}\,\boldsymbol{\varepsilon}\cdot\delta\mathbf{X} + Q\,, \tag{E.3}$$

where the total current Q is given by $Q = Q_{\mathbf{X}} + (Q_{\psi_1} + Q_{\psi_2} + c.c.)$. When the Euler-Lagrange equations are satisfied, $\delta H_c = Q$.

Consider now a patch of surface bounded by a set of closed curves. To determine the change in the equilibrium energy under the translation of one of these curves (say Γ), consider a deformation $\delta\mathbf{X}$ that reduces to a translation $\delta\mathbf{a}$ on this curve and vanishes on the remaining boundaries. The spinor field

transforms trivially under translation. Making use of the Gauss theorem the change in the energy can be recast as

$$\delta H_c = -\delta \mathbf{a} \cdot \mathbf{F}\,, \tag{E.4}$$

where the vector $\mathbf{F}$ is defined by the line integral

$$\mathbf{F} = \int_\Gamma ds\, l_z \mathbf{f}^z + c.c.\,. \tag{E.5}$$

with ds the line element along Γ, l_z and its c.c. are the components of the covector associated with the vector of the Darboux frame adapted to Γ and which is normal to it but tangent to the surface. Eq.(E.4) identifies the vector $\mathbf{F}$ as the force acting on the boundary curve Γ [24]. Furthermore, the quantity $l_z \mathbf{f}^z$ represents the local force acting on the line element ds, so that the conserved current associated with translational invariance $\mathbf{f}^z$ is correctly identified as the surface stress tensor.

Rotational invariance is a little more involved due to the non-trivial transformation properties of the spinor field. Using the identity (13) for the normal vector $\mathbf{n}$ in terms of the spinor field, it is easy to see that the variation in the spinor field induces a variation in $\mathbf{n}$ given by

$$\delta \mathbf{n} = \frac{2}{|\psi|^4}\left((\bar\psi_1 \delta\psi_2 - \psi_2 \delta\bar\psi_1)\mathbf{e}_z + (\psi_1 \delta\bar\psi_2 - \bar\psi_2 \delta\psi_1)\mathbf{e}_{\bar z}\right). \tag{E.6}$$

Making use of expressions (33) for λ and (E.6) for $\delta\mathbf{n}$ in the expressions for the currents given above, it is easily seen that Q is given by

$$Q = -i \int dz \wedge d\bar z \partial_z \left(\mathbf{f}^z \cdot \delta \mathbf{X} + \mathbf{c}^z \cdot \delta \mathbf{n}\right) + c.c.\,, \tag{E.7}$$

where $\mathbf{c}^z = \frac{1}{2\,|\psi|^2}\frac{\delta L}{\delta \mathcal{V}}\mathbf{e}_{\bar z}$.

Considering now a infinitesimal rotation by a constant angle $\delta\boldsymbol{\omega}$. The embedding functions transform by $\delta\mathbf{X} = \delta\boldsymbol{\omega} \times \mathbf{X}$; the normal vector also rotates, $\delta\mathbf{n} = \delta\boldsymbol{\omega} \times \mathbf{n}$. Thus the energy changes by

$$\delta H_c = -i\delta\boldsymbol{\omega} \cdot \int dz \wedge d\bar z\, \partial_z \mathbf{m}^z + c.c.\,, \tag{E.8}$$

where $\mathbf{m}^z = \mathbf{X} \times \mathbf{f}^z + 1/2|\psi|^{-2}\delta L/\delta\mathcal{V}\mathbf{n} \times \mathbf{e}_{\bar z}$. Using an identical argument to the one used to identify the force on the boundary curve Γ, we identify the change in the energy associated with a rotation of this curve

$$\delta H_c = -\delta\boldsymbol{\omega} \cdot \mathbf{M}\,, \tag{E.9}$$

where

$$\mathbf{M} = \int_\Gamma ds l_z \mathbf{m}^z + c.c.\,. \tag{E.10}$$

We thus identify $\mathbf{M}$ as the total torque acting on this boundary. $\mathbf{m}^z$ and its c.c. are identified as the components of the surface torque tensor [33]. Using the identity $\mathbf{e}_z = i\mathbf{e}_z \times \mathbf{n}$, the component of the torque tensor can be rewritten in the form

$$\mathbf{m}^z = \mathbf{X} \times \mathbf{f}^z - \frac{i}{2\,|\psi|^2}\frac{\delta L}{\delta \mathcal{V}}\mathbf{e}_{\bar z}\,. \tag{E.11}$$

Appendix F Consistency of Euler-Lagrange equations

At the end of section 2 it was noted that the Canham-Helfrich bending energy differs from the conformally invariant Willmore energy H_3, defined by Eq.(22) by a topological energy proportional to the Gauss-Bonnet invariant. The corresponding Euler-Lagrange equations therefore coincide; indeed, the corresponding local stresses also coincide. It is instructive to demonstrate this explicitly using the functional form of H_3 in terms of $|\mathcal{A}|$.

In this case derivatives $\partial_z \psi_1$ and $\partial_z \bar\psi_2$ appear explicit through their Wronskian $\mathcal{A}$ and its c.c.. The component ${f^z}_{\bar z}$ given by Eq.(46) is replaced by

$${f^z}_{\bar z} \to {f^z}_{\bar z} + \mathcal{V}\left(\psi_2 \frac{\partial L}{\partial(\partial_z \psi_1)} - \bar\psi_1 \frac{\partial L}{\partial(\partial_z \bar\psi_2)}\right). \tag{E.1}$$

For a functional which depends on derivatives of ψ no higher than first, the relevant components of the stress tensor are

$$
\begin{aligned}
f^z{}_z &= -\frac{1}{2}\left(\psi_1\left(\frac{\partial L}{\partial \psi_1} - \partial_z\left(\frac{\partial L}{\partial(\partial_z \psi_1)}\right)\right) + \bar{\psi}_2\left(\frac{\partial L}{\partial \bar{\psi}_2} - \partial_z\left(\frac{\partial L}{\partial(\partial_z \bar{\psi}_2)}\right)\right)\right), && \text{(E.2a)}\\
f^z{}_{\bar{z}} &= \frac{\bar{\mathcal{A}}}{2\,|\psi|^2}\frac{\partial L}{\partial \mathcal{V}} + \mathcal{V}\left(\psi_2\frac{\partial L}{\partial(\partial_z\psi_1)} - \bar{\psi}_1\frac{\partial L}{\partial(\partial_z\bar{\psi}_2)}\right), && \text{(E.2b)}\\
f^z &= -\partial_{\bar{z}}\left(\frac{1}{2|\psi|^2}\frac{\partial L}{\partial \mathcal{V}}\right) + \frac{1}{|\psi|^2}\left(\psi_2\left(\frac{\partial L}{\partial \psi_1} - \partial_z\left(\frac{\partial L}{\partial(\partial_z\psi_1)}\right)\right) - \bar{\psi}_1\left(\frac{\partial L}{\partial\bar{\psi}_2} - \partial_z\left(\frac{\partial L}{\partial(\partial_z\bar{\psi}_2)}\right)\right)\right) && \text{(E.2c)}
\end{aligned}
$$

Note that for functionals involving higher order derivatives, the corresponding functional derivatives will contain extra terms arising from integration by parts.

The required partial derivatives are

$$
\frac{\partial L}{\partial \psi_1} = -\frac{4}{|\psi|^4}\left(\bar{\mathcal{A}}\partial_z\bar{\psi}_2 + 2\frac{|\mathcal{A}|^2}{|\psi|^2}\bar{\psi}_1\right), \qquad \frac{\partial L}{\partial(\partial_z\psi_1)} = \frac{4}{|\psi|^4}\bar{\mathcal{A}}\bar{\psi}_2, \qquad \text{(E.3a)}
$$

$$
\frac{\partial L}{\partial \bar{\psi}_2} = \frac{4}{|\psi|^4}\left(\bar{\mathcal{A}}\partial_z\psi_1 - 2\frac{|\mathcal{A}|^2}{|\psi|^2}\psi_2\right), \qquad \frac{\partial L}{\partial(\partial_z\bar{\psi}_2)} = -\frac{4}{|\psi|^4}\bar{\mathcal{A}}\psi_1\,. \qquad \text{(E.3b)}
$$

Making use of the Codazzi-Mainardi Eq.(D.3b) we have that the corresponding functional derivatives are

$$
\frac{\delta\tilde{L}}{\delta\psi_1} = -\bar{\psi}_2\partial_{\bar{z}}K, \qquad \frac{\delta\tilde{L}}{\delta\bar{\psi}_2} = \psi_1\partial_{\bar{z}}K, \qquad \frac{\delta\tilde{L}}{\delta\mathcal{V}} = 0\,. \qquad \text{(E.4)}
$$

Substituting these identities into expressions (42) and (44) for the components of the stress tensor reproduces a stress identical to that for the Canham-Helfrich energy written in Eq.(62). It is worth pointing out that the two stress tensors did not need to coincide: they could have differed by a null stress.

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